CET 1:
Stress Analysis &
Pressure Vessels

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Supporting Materials

There is one Examples paper supporting these lectures.

Two good textbooks for further explanation, worked examples and exercises are


This material was taught in the CET I (Old Regulations) Structures lecture unit and was examined in CET I (OR) Paper IV Section 1. There are consequently a large number of old Tripos questions in existence, which are of the appropriate standard. From 1999 onwards the course was taught in CET1, paper 5. Chapters 7 and 8 in Gere and Timoshenko contain a large number of example problems and questions.

Nomenclature

The following symbols will be used as consistently as possible in the lectures.

- $E$ Young’s modulus
- $G$ Shear modulus
- $I$ second moment of area
- $J$ polar moment of area
- $R$ radius
- $t$ thickness
- $T$ τορθυε
- $\alpha$ thermal expansivity
- $\varepsilon$ linear strain
- $\gamma$ shear strain
- $\eta$ angle
- $\nu$ Poisson’s ratio
- $\sigma$ Normal stress
- $\tau$ Shear stress
A pressure vessel near you!
Ongoing Example

We shall refer back to this example of a typical pressure vessel on several occasions.

Distillation column

\[ P = 7 \text{ bara} \]
\[ \text{carbon steel} \]
\[ t = 5 \text{ mm} \]
1. Introduction to Pressure Vessels and Failure Modes

Pressure vessels are very often
- spherical (e.g. LPG storage tanks)
- cylindrical (e.g. liquid storage tanks)
- cylindrical shells with hemispherical ends (e.g. distillation columns)

Such vessels fail when the stress state somewhere in the wall material exceeds some failure criterion. It is thus important to be able to understand and quantify stresses in solids. This unit will concentrate on the application of stress analysis to bulk failure in thin walled vessels only, where (i) the vessel self weight can be neglected and (ii) the thickness of the material is much smaller than the dimensions of the vessel ($D \gg t$).

1.1. Stresses in Cylinders and Spheres

Consider a cylindrical pressure vessel

The hydrostatic pressure causes stresses in three dimensions.
1. Longitudinal stress (axial) $\sigma_L$
2. Radial stress $\sigma_r$
3. Hoop stress $\sigma_h$
all are normal stresses.
a. The longitudinal stress $\sigma_L$

Force equilibrium

$$\frac{\pi D^2}{4} P = \pi D t \sigma_L$$

if $P > 0$, then $\sigma_L$ is tensile

$$\sigma_L = \frac{P D}{4 t}$$

b. The hoop stress $\sigma_h$

Force balance, $D L P = 2 \sigma_h L t$

$$\sigma_h = \frac{P D}{2 t}$$
c, Radial stress

\[ \sigma_r \text{ varies from } P \text{ on inner surface to } 0 \text{ on the outer face} \]

\[ \sigma_r \approx o(P) \]

\[ \sigma_h, \sigma_L \approx P \left( \frac{D}{2t} \right). \]

thin walled, so \( D \gg t \)

so \( \sigma_h, \sigma_L >> \sigma_r \)

so neglect \( \sigma_r \)

Compare terms

d, The spherical pressure vessel

\[ P \frac{\pi D^2}{4} = \sigma_h \pi D t \]

\[ \sigma_h = \frac{PD}{4t} \]
1.2. Compressive Failure: – Bulk Yielding & Buckling – Vacuum Vessels

Consider an unpressurised cylindrical column subjected to a single load $W$. Bulk failure will occur when the normal compressive stress exceeds a yield criterion, e.g.

\[
\sigma_{bulk} = \frac{W}{\pi D t} = \sigma_Y
\]

Compressive stresses can cause failure due to buckling (bending instability).

The critical load for the onset of buckling is given by Euler’s analysis. A full explanation is given in the texts, and the basic results are summarised in the Structures Tables. A column or strut of length $L$ supported at one end will buckle if

\[
W = \frac{\pi^2 EI}{L^2}
\]

Consider a cylindrical column. $I = \pi R^3 t$ so the compressive stress required to cause buckling is

\[
\sigma_{buckle} = \frac{W}{\pi D t} = \frac{\pi^2 E \pi D^3 t}{8 L^2} \cdot \frac{1}{\pi D t} = \frac{\pi^2 E D^2}{8 L^2}
\]

or

\[
\sigma_{buckle} = \frac{\pi^2 E}{8(L/D)^2}
\]
where $L/D$ is a slenderness ratio. The mode of failure thus depends on the geometry:

![Graph showing the relationship between L/D ratio and stress](image)

- Euler buckling locus
- Bulk yield
- Short and Long L/D ratio
**Vacuum vessels.**
Cylindrical pressure vessels subject to external pressure are subject to compressive hoop stresses

\[ \sigma_h = \frac{\Delta P D}{2t} \]

Consider a length \( L \) of vessel, the compressive hoop force is given by,

\[ \sigma_h L t = \frac{\Delta P D L}{2} \]

If this force is large enough it will cause buckling.

Treat the vessel as an encasted beam of length \( \pi D \) and breadth \( L \)
Buckling occurs when Force $W$ given by:

$$W = \frac{4\pi^2EI}{(\pi D)^2}$$

and

$$\Delta P = \frac{4\pi^2EI}{2(\pi D)^2}$$

Moment $I$ is:

$$I = \frac{bt^3}{12} = \frac{Lt^3}{12}$$

and

$$\Delta p_{\text{buckle}} = \frac{2E}{3}\left(\frac{t}{D}\right)^3$$
1.3. Tensile Failure: Stress Concentration & Cracking

Consider the rod in the Figure below subject to a tensile load. The stress distribution across the rod a long distance away from the change in cross section (XX) will be uniform, but near XX the stress distribution is complex.

There is a concentration of stress at the rod surface below XX and this value should thus be considered when we consider failure mechanisms.

The ratio of the maximum local stress to the mean (or apparent) stress is described by a stress concentration factor $K$

$$K = \frac{\sigma_{\text{max}}}{\sigma_{\text{mean}}}$$

The values of $K$ for many geometries are available in the literature, including that of cracks. The mechanism of fast fracture involves the concentration of tensile stresses at a crack root, and gives the failure criterion for a crack of length $a$

$$\sigma\sqrt{\pi a} = K_c$$

where $K_c$ is the material fracture toughness. Tensile stresses can thus cause failure due to bulk yielding or due to cracking.
\[ \sigma_{\text{crack}} = \frac{K_c}{\sqrt{\pi a}} \]
2. 3-D stress and strain

2.1. Elasticity and Yield

Many materials obey Hooke's law

\[
\sigma = E \varepsilon
\]

- \(\sigma\): applied stress (Pa)
- \(E\): Young's modulus (Pa)
- \(\varepsilon\): strain (-)

up to a limit, known as the yield stress (stress axis) or the elastic limit (strain axis). Below these limits, deformation is reversible and the material eventually returns to its original shape. Above these limits, the material behaviour depends on its nature.

Consider a sample of material subjected to a tensile force \(F\).

An increase in length (axis 1) will be accompanied by a decrease in dimensions 2 and 3.

Hooke's Law

\[
\varepsilon_1 = \left(\frac{\sigma_1}{E} = \frac{F}{A}\right) / E
\]
The strain in the perpendicular directions 2,3 are given by

\[ \varepsilon_2 = -\nu \frac{\sigma_1}{E}; \varepsilon_3 = -\nu \frac{\sigma_1}{E} \]

where \( \nu \) is the Poisson ratio for that material. These effects are additive, so for three mutually perpendicular stresses \( \sigma_1, \sigma_2, \sigma_3 \);

\[ \varepsilon_1 = \frac{\sigma_1}{E} - \nu \frac{\sigma_2}{E} - \nu \frac{\sigma_3}{E} \]
\[ \varepsilon_2 = -\nu \frac{\sigma_1}{E} + \frac{\sigma_2}{E} - \nu \frac{\sigma_3}{E} \]
\[ \varepsilon_3 = -\nu \frac{\sigma_1}{E} - \nu \frac{\sigma_2}{E} + \frac{\sigma_3}{E} \]

Values of the material constants in the Data Book give orders of magnitudes of these parameters for different materials;

<table>
<thead>
<tr>
<th>Material</th>
<th>( E ) (x10^9 N/m²)</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>210</td>
<td>0.30</td>
</tr>
<tr>
<td>Aluminum alloy</td>
<td>70</td>
<td>0.33</td>
</tr>
<tr>
<td>Brass</td>
<td>105</td>
<td>0.35</td>
</tr>
</tbody>
</table>
2.2 Bulk and Shear Moduli

These material properties describe how a material responds to an applied stress (bulk modulus, $K$) or shear (shear modulus, $G$).

The bulk modulus is defined as

\[ P_{\text{uniform}} = -K\varepsilon_v \]

i.e. the volumetric strain resulting from the application of a uniform pressure. In the case of a pressure causing expansion

\[ \sigma_1 = \sigma_2 = \sigma_3 = -P \]

so

\[ \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{E} \left[ \sigma_1 - \nu \sigma_2 - \nu \sigma_3 \right] = -\frac{P}{E} (1 - 2\nu) \]

\[ \varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = -\frac{3P}{E} (1 - 2\nu) \]

\[ K = \frac{E}{3(1 - 2\nu)} \]

For steel, $E = 210$ kN/mm$^2$, $\nu = 0.3$, giving $K = 175$ kN/mm$^2$

For water, $K = 2.2$ kN/mm$^2$

For a perfect gas, $K = P$ (1 bara, $10^{-4}$ kN/mm$^2$)

Shear Modulus definition

\[ \tau = G\gamma \]

$\gamma$ - shear strain
2.3. Hoop, longitudinal and volumetric strains
(micro or millistrain)

Fractional increase in dimension:

\[ \varepsilon_L - \text{length} \]

\[ \varepsilon_h - \text{circumference} \]

\[ \varepsilon_r - \text{wall thickness} \]

(a) Cylindrical vessel:

Longitudinal strain

\[ \varepsilon_L = \frac{\sigma_L}{E} - \frac{\nu \sigma_h}{E} - \frac{\nu \sigma_r}{E} = \frac{PD}{4tE}(1 - 2\nu) = \frac{\delta L}{L} \]

Hoop strain:

\[ \varepsilon_h = \frac{\sigma_h}{E} - \frac{\nu \sigma_L}{E} = \frac{PD}{4tE}(2 - \nu) = \frac{\delta D}{D} = \frac{\delta R}{R} \]

Radial strain

\[ \varepsilon_r = \frac{1}{E} \left[ \sigma_r - \nu \sigma_h - \nu \sigma_L \right] = -\frac{3PD\nu}{4ET} = \frac{\delta t}{t} \]

[fractional increase in wall thickness is negative!]
[ONGOING EXAMPLE]:

\[
\varepsilon_L = \frac{1}{E} (\sigma_L - \nu \sigma_n)
\]

\[
= \frac{1}{210 \times 10^9} [60 \times 10^6 - (0.3)120 \times 10^6]
\]

\[
= 1.14 \times 10^{-4} = 0.114 \text{ millistrain}
\]

\(\varepsilon_h = 0.486 \text{ millistrain}\)

\(\varepsilon_r = -0.257 \text{ millistrain}\)

Thus: pressurise the vessel to 6 bar: \(L\) and \(D\) increase; \(t\) decreases

**Volume expansion**

**Cylindrical volume:**

\[
V_o = \left( \frac{\pi D_o^2}{4} \right) L_o \quad \text{(original)}
\]

New volume

\[
V = \frac{\pi}{4} (D_o + \delta D)^2 (L_o + \delta L)
\]

\[
= \frac{L_o \pi D_o^2}{4} [1 + \varepsilon_h]^2 [1 + \varepsilon_L]
\]

Define volumetric strain \(\varepsilon_v = \frac{\delta V}{V}\)

\[
\therefore \varepsilon_v = \frac{V - V_o}{V_o} = (1 + \varepsilon_h)^2 (1 + \varepsilon_L) - 1
\]

\[
= (1 + 2\varepsilon_h + \varepsilon_h^2)(1 + \varepsilon_L) - 1
\]

\[
\varepsilon_v = 2\varepsilon_h + \varepsilon_L + \varepsilon_h^2 + 2\varepsilon_h \varepsilon_L + \varepsilon_L \varepsilon_h^2
\]

Magnitude inspection:
\[ \varepsilon_{\text{max (steel)}} = \frac{\sigma_y}{E} = \frac{190 \times 10^6}{210 \times 10^9} = 0.905 \times 10^{-3} \therefore \text{small} \]

Ignoring second order terms,

\[ \varepsilon_v = 2\varepsilon_h + \varepsilon_L \]

(b) Spherical volume:

\[ \varepsilon_h = \frac{1}{E} \left[ \sigma_h - \nu \sigma_L - \nu \sigma_r \right] = \frac{PD}{4Et} (1 - \nu) \]

so

\[ \varepsilon_v = \frac{\pi}{6} \frac{(D_o + \delta D)^3 - D_o^3}{\pi D_o^3 / 6} \]

\[ = (1 + \varepsilon_h)^3 - 1 \approx 3\varepsilon_h + 0(\varepsilon^2) \]

(c) General result

\[ \varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \]

\( \varepsilon_{ii} \) are the strains in any three mutually perpendicular directions.

{Continued example} – cylinder

\[ \varepsilon_L = 0.114 \text{ mstrain} \]

\[ \varepsilon_n = 0.486 \text{ mstrain} \]

\[ \varepsilon_{rr} = -0.257 \text{ mstrain} \]

\[ \varepsilon_v = 2\varepsilon_n + \varepsilon_L \]

\[ \therefore \text{new volume} = V_o (1 + \varepsilon_v) \]

Increase in volume = \[ \frac{\pi D^2 L}{4} \varepsilon_v = 56.55 \times 1.086 \times 10^{-3} \]

\[ = 61 \text{ Litres} \]

Volume of steel\_o = \[ \pi DLt = 0.377 \text{ m}^3 \]

\[ \varepsilon_v \text{ for steel} = \varepsilon_L + \varepsilon_h + \varepsilon_{rr} = 0.343 \text{ mstrains} \]

increase in volume of steel \[ = 0.129 \text{ L} \]

Strain energy – measure of work done

Consider an elastic material for which \( F = k x \)
Work done in expanding $\delta x$

$$dW = F \delta x$$

Work done in extending to $x_1$

$$w = \int_0^{x_1} Fdx = \int_0^{x_1} kx \, dx = \frac{kx_1^2}{2} = \frac{1}{2} F_1 x_1$$

Sample subject to stress $\sigma$ increased from 0 to $\sigma_1$:

Extension Force:

$$\begin{align*}
\frac{x_1}{F_1} &= L_0 \varepsilon_1 \\
W &= \frac{AL_0 \varepsilon_1 \sigma_1}{2}
\end{align*}$$

(no direction here)

Strain energy, $U$ = work done per unit volume of material, $U = \frac{AL_0 \varepsilon_1 \sigma_1}{2(AL_0)}$

$$\Rightarrow U = \frac{Al_0}{2} \left( \frac{1}{Al_0} \right) \varepsilon_1 \sigma_1$$

$$U = \frac{\varepsilon_1 \sigma_1}{2} = \frac{\sigma_1^2}{2E}$$

In a 3-D system, $U = \frac{1}{2} \left[ \varepsilon_1 \sigma_1 + \varepsilon_2 \sigma_2 + \varepsilon_3 \sigma_3 \right]$}

Now $\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\nu \sigma_2}{E} - \frac{\nu \sigma_3}{E}$ etc

So $U = \frac{1}{2E} \left[ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2 \nu (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1) \right]$}

Consider a uniform pressure applied: $\sigma_1 = \sigma_2 = \sigma_3 = P$

$$\therefore U = \frac{3P^2}{2E} \left( 1 - 2\nu \right) = \frac{P^2}{2K} \text{ energy stored in system (per unit vol.}$$
For a given $P$, $U$ stored is proportional to $1/K \rightarrow$ so pressure test using liquids rather than gases.

{Ongoing Example} \hspace{1cm} P = 6 \text{ barg} \hspace{1cm} \delta V = 61 \times 10^{-3} \text{ m}^3

*increase* in volume of pressure vessel

Increasing the pressure compresses the contents – normally test with water.

$\Delta V \text{ water? } \varepsilon_v \text{(water)} = -\frac{\Delta P}{K} = - \frac{6 \times 10^5}{2.2 \times 10^9} = -0.273 \text{ mstrains}$

$\therefore$ decrease in volume of water $= -V_0 (0.273) = -15.4 \times 10^{-3} \text{ m}^3$

Thus we can add more water:

Extra space $= 61 + 15.4 \text{ (L)}$

$= 76.4 \text{ L water}$

```
p=0
```

```
p=6
extra space
```
3. Thermal Effects

3.1. Coefficient of Thermal Expansion

**Definition:** coefficient of thermal expansion \( \varepsilon = \alpha L \Delta T \) Linear

Coefficient of thermal volume expansion \( \varepsilon_v = \alpha T \) Volume

**Steel:** \( \alpha_L = 11 \times 10^{-6} \text{ K}^{-1} \)

- \( \Delta T = 10^\circ C \) \( \therefore \varepsilon_L = 11 \times 10^{-5} \)
- reactor \( \Delta T = 500^\circ C \) \( \varepsilon_L = 5.5 \text{ millistrains (!) } \)

Consider a steel bar mounted between rigid supports which exert stress \( \sigma \)

\[
\varepsilon = \alpha \Delta T - \frac{\sigma}{E}
\]

If rigid: \( \varepsilon = 0 \) \( \Rightarrow \) so \( \sigma = E \alpha \Delta T \)

(i.e., non buckling)

**Steel:** \( \sigma = 210 \times 10^9 \times 11 \times 10^{-6} \Delta T = 2.3 \times 10^6 \Delta T \)

\( \sigma_y = 190 \text{ MPa: failure if } \Delta T > 82.6 \text{ K } \)
Example: steam main, installed at 10°C, to contain 6 bar steam (140°C)
if ends are rigid, $\sigma = 300$ MPa $\rightarrow$ failure.
$\therefore$ must install expansion bends.

3.2. Temperature effects in cylindrical pressure vessels

![Diagram of cylindrical pressure vessel]

D = 1 m steel construction L = 3 m full of water t = 3 mm
Initially un pressurised – full of water: increase temp. by $\Delta T$: pressure rises to Vessel P.

The Vessel Wall stresses (tensile) $\sigma_L = \frac{PD}{4t} = 83.3$ P

$\sigma_n = 2\sigma_L = 166.7$ P
**Strain (volume)**

\[
\varepsilon_L = \frac{\sigma_L}{E} - \frac{\nu\sigma_h}{E} + \alpha_1 \Delta T
\]

\[\begin{align*}
\nu &= 0.3 \\
E &= 210 \times 10^9 \\
\alpha &= 11 \times 10^{-6}
\end{align*}\]

\[\varepsilon_L = 1.585 \times 10^{-10} P + 11 \times 10^{-6} \Delta T\]

Similarly

\[\varepsilon_h = 6.75 \times 10^{-10} P + 11 \times 10^{-6} \Delta T\]

\[\varepsilon_v = \varepsilon_L + 2\varepsilon_n = 15.08 \times 10^{-10} P + 33 \times 10^{-6} \Delta T = \text{vessel vol. Strain}\]

vessel expands due to temp and pressure change.

**The Contents, (water)**

**Expands** due to T increase

**Contracts** due to P increase:

\[\varepsilon_v, \text{H}_2\text{O} = \alpha_v \Delta T - P/K\]

\[\text{H}_2\text{O} = \alpha_v = 60 \times 10^{-6} \text{K}^{-1}\]

\[\therefore \varepsilon_v = 60 \times 10^{-6} \Delta T - 4.55 \times 10^{-10} P\]

Since vessel remains full on increasing \(\Delta T\):

\[\varepsilon_v (\text{H}_2\text{O}) = \varepsilon_v (\text{vessel})\]

Equating \(\rightarrow P = 13750 \Delta T\)

pressure, rise of 1.37 bar per 10°C increase in temp.

Now

\[\sigma_n = 166.7 \text{ P} = 2.29 \times 10^6 \Delta T\]

\[\Delta\sigma_n = 22.9 \text{ Mpa per} 10^\circ \text{C rise in Temperature}\]
Failure does not need a large temperature increase.
Very large stress changes due to temperature fluctuations.

MORAL: Always leave a space in a liquid vessel.

\((\varepsilon_v, \text{gas} = \alpha_v \Delta T - P/K)\)
3.3. Two material structures

Beware, different materials with different thermal expansivities can cause difficulties.

{Example} Where there is benefit. **The Bimetallic strip** - temperature controllers

\[a = 4\text{ mm (2 + 2 mm)} \quad b = 10\text{ mm}\]

Heat by \(\Delta T\): Cu expands more than Fe so the strip will bend: it will bend in an arc as all sections are identical.
The different thermal expansions, set up shearing forces in the strip, which create a bending moment. If we apply a sagging bending moment of equal magnitude, we will obtain a straight beam and can then calculate the shearing forces [and hence the BM].

Shearing force $F$ compressive in Cu
Tensile in Fe

Equating strains:

$$\alpha_{cu} \Delta T - \frac{F}{bdE_{cu}} = \alpha_{Fe} \Delta T + \frac{F}{bdE_{Fe}}$$

So

$$\frac{F}{bd} \left\{ \frac{1}{E_{cu}} + \frac{1}{E_{Fe}} \right\} = (\alpha_{cu} - \alpha_{Fe}) \Delta T$$
\[ \bar{bd} = 2 \times 10^{-5} \text{ m}^2 \]

\[ E_{cu} = 109 \text{ GPa} \quad \alpha_{cu} = 17 \times 10^{-6} \text{ k}^{-1} \quad \Delta T = 30^\circ \text{C} \quad F = 387 \text{ N} \]

\[ E_{Fe} = 210 \text{ GPa} \quad \alpha_{Fe} = 11 \times 10^{-6} \text{ K}^{-1} \quad \text{(significant force)} \]

F acts through the centroid of each section so BM = \( F.d/(d/2) \) = 0.774 Nm

Use data book to work out deflection.

\[ \delta = \frac{ML^2}{2EI} \]

This is the principle of the bimetallic strip.
Consider a steel rod mounted in a upper tube – spacer

Analysis – relevant to Heat Exchangers

![Diagram of a steel rod mounted in a copper tube](image)

assembled at room temperature

increase $\Delta T$

Data: $\alpha_{\text{cu}} > \alpha_{\text{Fe}}$ : copper expands more than steel, so will generate a TENSILE stress in the steel and a compressive stress in the copper.

Balance forces:

Tensile force in steel $|F_{\text{Fe}}| = |F_{\text{cu}}| = F$

Stress in steel $= F/A_{\text{Fe}} = \sigma_{\text{Fe}}$

“ “ copper $= F/A_{\text{cu}} = \sigma_{\text{cu}}$

Steel strain: $\varepsilon_{\text{Fe}} = \alpha_{\text{Fe}} \Delta T + \sigma_{\text{Fe}}/E_{\text{Fe}}$ (no transverse forces)

$= \alpha_{\text{Fe}} \Delta T + F/E_{\text{Fe}}A_{\text{Fe}}$

Copper strain $\varepsilon_{\text{cu}} = \alpha_{\text{cu}} \Delta T - F/E_{\text{cu}}A_{\text{cu}}$
Strains EQUAL: \[ F \left[ \frac{1}{A_{Fe}E_{Fe}} + \frac{1}{A_{cu}E_{cu}} \right] = \left( \alpha_{cu} - \alpha_{Fe} \right) \Delta T \]

So you can work out stresses and strains in a system.
4. Torsion – Twisting – Shear stresses

4.1. Shear stresses in shafts – $\tau/r = T/J = G \theta/L$

Consider a rod subject to twisting:

Definition: shear strain $\gamma \equiv$ change in angle that was originally $\Pi/2$

Consider three points that define a right angle and more then:

Shear strain $\gamma = \gamma_1 + \gamma_2$ [RADIANS]

Hooke’s Law $\tau = G \gamma$

$G$ – shear modulus $= \frac{E}{2(1 + \nu)}$
Now consider a rod subject to an applied torque, $T$.

Hold one end and rotate other by angle $\theta$

Plane ABO was originally to the X-X axis

Plane ABO is now inclined at angle $\gamma$ to the axis: $\tan \gamma \approx \gamma = \frac{r\theta}{L}$

Shear stress involved $\tau = G\gamma = \frac{Gr\theta}{L}$
Torque required to cause twisting:

\[ \delta T = \tau \ 2\Pi \ r \ \delta r \ r \]

\[ T = \int_A \tau \ r \ dA \quad \text{or} \quad \int_A \tau \ r \ dA \quad \left\{ \tau = \frac{Gr\theta}{L} \right\} \]

\[ T = \frac{G\theta}{L} \int r^2 dA \]

\[ = \frac{G\theta}{L} \{J\} \quad \text{so} \quad \frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{r} \]

cf \[ \frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y} \]

DEFN: \( J \equiv \text{polar second moment of area} \)
Consider a rod of circular section:

\[ J = \int_0^R 2\pi r^2 \, dr = \frac{\pi}{2} R^4 \]

Now

\[ r^2 = x^2 + y^2 \]

It can be shown that \( J = I_{xx} + I_{yy} \) [perpendicular axis than] → see Fenner. This gives an easy way to evaluate \( I_{xx} \) or \( I_{yy} \) in symmetrical geometries:

\[ I_{xx} = I_{yy} = \frac{\pi D^4}{64} \text{ (rod)} \]
Rectangular rod:

\[
I_{xx} = \frac{bd^3}{12} \\
I_{yy} = \frac{db^3}{12} \\
J = \frac{bd}{12} \left( b^2 + d^2 \right)
\]
Example: steel rod as a drive shaft

D = 25 mm
L = 1.5 m
Failure when $\tau = \tau_y = 95$ MPa
G = 81 GPa

\[
\frac{\tau_{\text{max}}}{r_{\text{max}}} = \frac{T}{J} = \frac{95 \times 10^6}{0.0125}
\]

Now
\[
J = \frac{\pi D^4}{32} = 383 \times 10^{-8} \text{ m}^4
\]

so $T = 291$ Nm

\[
\text{From } \frac{G \theta}{L} = \frac{T}{J} \Rightarrow 7.6 \times 10^3 = \frac{(81 \times 10^9) \theta}{1.5} \Rightarrow \theta = 0.141 \text{ rad} = 8.1^\circ
\]

Say shaft rotates at 1450 rpm: power $= T\omega$

$= 291 \times \frac{2\pi}{60} \times 1450$

$= 45$ kW
4.2. Thin walled shafts

(same Eqns apply)

Consider a bracket joining two Ex. Shafts:

\[ T = 291 \text{ Nm} \]

\[ r_{\text{max}} = \frac{D}{2} \]

\[ J = \left( \frac{\pi}{32} \right) \{ D^4 - 0.025^4 \} \]

\[ \frac{\tau_y}{r_{\text{max}}} = \left( \frac{T}{J} \right) \Rightarrow \frac{95 \times 10^6 \times 2}{D} = \frac{291}{\pi} \frac{32}{D^4 - 0.025^4} \]

\[ D^4 - 0.025^4 = 6.24 \times 10^{-5} D \quad D \geq 4.15 \text{ cm} \]
4.3. Thin walled pressure vessel subject to torque

\[ \tau = \frac{T}{r} \quad \text{now cylinder} \quad J = \frac{\pi}{32} \left[ (D + 2t)^4 - D^4 \right] \]

\[ = \frac{\pi}{32} \left[ 8D^3 t + 24 D^2 t^2 + \ldots \right] \]

\[ \approx \frac{\pi}{4} D^3 t \]

so \[ \frac{\tau}{D} = \frac{4T}{\pi D^3 t} \]

\[ \tau = \frac{2T}{\pi D^2 t} \]
5. Components of Stress/ Mohr’s Circle

5.1 Definitions

Scalars

tensor of rank 0

Vectors

tensors of rank 1

\[ \vec{F} = m \vec{a} \]

hence:

\[ F_1 = ma_1 \]
\[ F_2 = ma_2 \]
\[ F_3 = ma_3 \]

or:

\[ F_i = ma_i \]

Tensors of rank 2

\[ p_i = \sum_{j=1}^{3} T_{ij} q_j \quad i, j = 1,2,3 \]

or:

\[ p_1 = T_{11} q_1 + T_{12} q_2 + T_{13} q_3 \]
\[ p_2 = T_{21} q_1 + T_{22} q_2 + T_{23} q_3 \]
\[ p_3 = T_{31} q_1 + T_{32} q_2 + T_{33} q_3 \]

Axis transformations

The choice of axes in the description of an engineering problem is arbitrary (as long as you choose orthogonal sets of axes!). Obviously the physics of the problem must not depend on the choice of axis. For example, whether a pressure vessel will explode can not depend on how we set up our co-ordinate axes to describe the stresses acting on the
vessel. However it is clear that the components of the stress tensor will be different going from one set of coordinates \( x_i \) to another \( x_i' \).

How do we transform one set of co-ordinate axes onto another, keeping the same origin?

\[
\begin{array}{ccc}
  x_1 & x_2 & x_3 \\
  x_1' & a_{11} & a_{12} & a_{13} \\
  x_2' & a_{21} & a_{22} & a_{23} \\
  x_3' & a_{31} & a_{32} & a_{33} \\
\end{array}
\]

... where \( a_{ij} \) are the direction cosines

Forward transformation:
\[
x_i' = \sum_{j=1}^{3} a_{ij} x_j \\
\text{“New in terms of Old”}
\]

Reverse transformation:
\[
x_i = \sum_{j=1}^{3} a_{ji} x_j \\
\text{“Old in terms of New”}
\]

We always have to do summations in co-ordinate transformation and it is conventional to drop the summation signs and therefore these equations are simply written as:

\[
x_i' = a_{ij} x_j
\]

\[
x_i = a_{ji} x_j
\]
**Tensor transformation**

How will the components of a tensor change when we go from one co-ordinate system to another? I.e. if we have a situation where

\[ p_i = \sum_j T_{ij} q_j = T_{ij} q_j \text{ (in short form)} \]

where \( T_{ij} \) is the tensor in the old co-ordinate frame \( x_i \), how do we find the corresponding tensor \( T_{ij}' \) in the new co-ordinate frame \( x_i' \), such that:

\[ p_i' = \sum_j T_{ij}' q_j' = T_{ij}' q_j' \text{ (in short form)} \]

We can find this from a series of sequential co-ordinate transformations:

\[ p' \leftarrow p \leftarrow q \leftarrow q' \]

Hence:

\[ p_i' = a_{ik} p_k \]

\[ p_k = T_{kl} q_l \]

\[ q_l = a_{jl} q_j' \]

Thus we have:
\[ p_i' = a_{ik} T_{kl} a_{jl} q_j' \]

\[ = a_{ik} a_{jl} T_{kl} q_j' \]

\[ = T_{ij}' q_j' \]

For example:

\[ T_{ij}' = a_{i1} a_{j1} T_{1l} \]

\[ + a_{i2} a_{j1} T_{2l} \]

\[ + a_{i3} a_{j1} T_{3l} \]

\[ = a_{i1} a_{j1} T_{11} + a_{i1} a_{j2} T_{12} + a_{i1} a_{j3} T_{13} \]

\[ + a_{i2} a_{j1} T_{21} + a_{i2} a_{j2} T_{22} + a_{i2} a_{j3} T_{23} \]

\[ + a_{i3} a_{j1} T_{31} + a_{i3} a_{j2} T_{32} + a_{i3} a_{j3} T_{33} \]

Note that there is a difference between a transformation matrix and a 2\textsuperscript{nd} rank tensor: They are both matrices containing 9 elements (constants) but:

**Symmetrical Tensors:**

\[ T_{ij} = T_{ji} \]
We can always transform a second rank tensor which is symmetrical:

\[ T_{ij} \rightarrow T_{ij}' \]

such that:

\[
T_{ij}' = \begin{bmatrix}
T_1 & 0 & 0 \\
0 & T_2 & 0 \\
0 & 0 & T_3
\end{bmatrix}
\]

Consequence? Consider:

\[ p_i = T_{ij} q_j \]

then

\[ p_1 = T_1 q_1, \quad p_2 = T_2 q_2, \quad p_3 = T_3 q_3 \]

The diagonal \( T_1, T_2, T_3 \) is called the PRINCIPAL AXIS.

If \( T_1, T_2, T_3 \) are stresses, then these are called PRINCIPAL STRESSES.
Mohr’s circle

Consider an elementary cuboid with edges parallel to the coordinate directions x,y,z.

The faces on this cuboid are named according to the directions of their normals.

There are thus two x-face, one facing greater values of x, as shown in Figure 1 and one facing lesser values of x (not shown in the Figure). On the x-face there will be some force $F_x$. Since the cuboid is of infinitesimal size, the force on the opposite side will not differ significantly.

The force $F_x$ can be divided into its components parallel to the coordinate directions, $F_{xx}, F_{xy}, F_{xz}$. Dividing by the area of the x-face gives the stresses on the x-plane:

$$\sigma_{xx}, \tau_{xy}, \tau_{xz}$$

It is traditional to write normal stresses as $\sigma$ and shear stresses as $\tau$.

Similarly, on the y-face: $\tau_{yx}, \sigma_{yy}, \tau_{yz}$
and on the z-face we have: $\tau_{xz}$, $\tau_{zy}$, $\sigma_{zz}$

There are therefore 9 components of stress;

$$\sigma_{ij} = \begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{bmatrix}$$

Note that the first subscript refers to the face on which the stress acts and the second subscript refers to the direction in which the associated force acts.

But for non accelerating bodies (or infinitesimally small cuboids):

and therefore:

$$\sigma_{ij} = \begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{bmatrix} \begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{bmatrix}$$

Hence $\sigma_{ij}$ is symmetric!
This means that there must be some magic co-ordinate frame in which all the stresses are normal stresses (principal stresses) and in which the off-diagonal stresses (=shear stresses) are 0. So if, in a given situation we find this frame we can apply all our stress strain relations that we have set up in the previous lectures (which assumed there were only normal stresses acting).

Consider a cylindrical vessel subject to shear, and normal stresses ($\sigma_h$, $\sigma_l$, $\sigma_r$). We are usually interested in shears and stresses which lie in the plane defined by the vessel walls.

Is there a transformation about $zz$ which will result in a shear

Would really like to transform into a co-ordinate frame such that all components in the $x_i'$:

$$\sigma_{ij} \rightarrow \sigma'_{ij}$$

So stress tensor is symmetric 2nd rank tensor. Imagine we are in the co-ordinate frame $x_i$ where we only have principal stresses:

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Transform to a new co-ordinate frame $x_i'$ by rotatoin about the $x_3$ axis in the original co-ordinate frame (this would be, in our example, $z$-axis)
The transformation matrix is then:

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
= 
\begin{pmatrix}
cos \theta & sin \theta & 0 \\
-sin \theta & cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Then:

\[
\sigma_{ij}' = a_{ik} \ a_{jl} \ \sigma_{kl}
\]

\[
\begin{pmatrix}
cos \theta & sin \theta & 0 \\
-sin \theta & cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
cos \theta & -sin \theta & 0 \\
sin \theta & cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{pmatrix}
\]

\[
\begin{pmatrix}
\sigma_1 \cos^2 \theta + \sigma_2 \sin^2 \theta & \sigma_1 \cos \theta \sin \theta + \sigma_2 \cos \theta \sin \theta & 0 \\
-\sigma_1 \cos \theta \sin \theta + \sigma_2 \cos \theta \sin \theta & \sigma_1 \cos^2 \theta + \sigma_2 \sin^2 \theta & 0 \\
0 & 0 & \sigma_3
\end{pmatrix}
\]
Hence:

\[ \sigma_{11}' = \sigma_1 \cos^2 \theta + \sigma_2 \sin^2 \theta \]

\[ = \frac{1}{2} (\sigma_1 + \sigma_1) - \frac{1}{2} (\sigma_2 - \sigma_1) \cos 2\theta \]

\[ \sigma_{22}' = \sigma_1 \sin^2 \theta + \sigma_2 \cos^2 \theta \]

\[ = \frac{1}{2} (\sigma_1 + \sigma_1) + \frac{1}{2} (\sigma_2 - \sigma_1) \cos 2\theta \]

\[ \sigma_{12}' = -\sigma_1 \cos \theta \sin \theta + \sigma_2 \cos \theta \sin \theta \]

\[ = \frac{1}{2} (\sigma_2 - \sigma_1) \sin 2\theta \]
Yield conditions. Tresca and Von Mises

Mohr's circle in three dimensions.
We can draw Mohrs circles for each principal plane.

6. BULK FAILURE CRITERIA

Materials fail when the largest stress exceeds a critical value. Normally we test a material in simple tension:

\[
\sigma_y = \frac{P_{\text{yield}}}{A}
\]

This material fails under the stress combination \((\sigma_y, 0, 0)\)

\[\tau_{\text{max}} = 0.5 \sigma_y = 95 \text{ Mpa} \text{ for steel}\]

We wish to establish if a material will fail if it is subject to a stress combination \((\sigma_1, \sigma_2, \sigma_3)\) or \((\sigma_n, \sigma_l, \tau)\)
Failure depends on the nature of the material:

Two important criteria

(i) **Tresca’s failure criterion**: brittle materials

   Cast iron: concrete: ceramics

(ii) **Von Mises’ criterion**: ductile materials

   Mild steel + copper

6.1. Tresca’s Failure Criterion; The Stress Hexagon (Brittle)

A material fails when the largest shear stress reaches a critical value, the yield shear stress $\tau_y$.

Case (i) Material subject to simple compression:

Principal stresses ($-\sigma_1$, 0, 0)
M.C: mc passes through \((\sigma_1,0), (\sigma_1,0), (0,0)\)

\[
\tau_{\text{max}} = \frac{\sigma_1}{2}
\]

occurs along plane at 45° to \(\sigma_1\)
Similarly for tensile test.

Case (ii) \(\sigma_2 < 0 < \sigma_1\)

\[
\text{M.C. Fails when } \frac{\sigma_1 - \sigma_2}{2} = \tau_{\text{max}} = \tau_y = \frac{\sigma_y}{2}
\]

i.e., when \(\sigma_1 - \sigma_2 = \sigma_y\)

material will not fail.
Let's do an easy example.
6.2 Von Mises’ Failure Criterion; The stress ellipse (ductile materials)

Tresca’s criterion does not work well for ductile materials. Early hypothesis – material fails when its strain energy exceeds a critical value (can’t be true as no failure occurs under uniform compression).

Von Mises’: failure when strain energy due to distortion, \( U_D \), exceeds a critical value.

\( U_D = \text{difference in strain energy (}U\text{) due of a compressive stress } C \text{ equal to the mean of the principal stresses.} \)

\[
C = \frac{1}{3} [\sigma_1 + \sigma_2 + \sigma_3]
\]

\[
U_D = \frac{1}{2E} \left[ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\nu (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) - \frac{1}{2E} [3C^2 + 6\nu C^2] \right]
\]

\[
= \frac{1}{12G} \left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\}
\]

M.C.
Tresca → failure when $\max (\tau_i) \geq \tau_y$  

Von Mises → failure when root mean square of $\{\tau_a, \tau_b, \tau_c\} \geq \text{critical value}$

$$\frac{1}{12G} \left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 + \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\} > \frac{1}{12G} \left\{ \sigma_y^2 + 0^2 + \sigma_y^2 \right\}$$

$$\left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 + \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\} > 2\sigma_y^2$$

Compare with $(\sigma_y, \sigma, \sigma)$ · simple tensile test – failure

Failure if
Lets do a simple example.
Example Tresca's Failure Criterion

The same pipe as in the first example \((D = 0.2 \text{ m}, t' = 0.005 \text{ m})\) is subject to an internal pressure of 50 barg. What torque can it support?

\[
\sigma_L = \frac{PD}{4t'} = 50 \text{ N/mm}^2
\]

Calculate stresses

\[
\sigma_h = \frac{PD}{2t'} = 100 \text{ N/mm}^2
\]

and \(\sigma_3 = \sigma_r \approx 0\)

Mohr's Circle

\[
s = 75 \text{ N/mm}^2
\]

\[
t = \sqrt{(25^2 + \tau^2)}
\]

The principal stresses \(\sigma_{1,2} = s \pm t\)

Thus \(\sigma_2\) may be positive (case A) or negative (case B). Case A occurs if \(\tau\) is small.
We do not know whether the Mohr’s circle for this case follows Case A or B; determine which case applies by trial and error.

Case A; 'minor' principal stress is positive ($\sigma_2 > 0$)

Thus failure when $\tau_{\text{max}} = \frac{1}{2} \sigma_y = 105 N/mm^2$
For Case A;

\[ \tau_{\text{max}} = \frac{\sigma_1}{2} = \frac{1}{2} \left[ 75 + \sqrt{25^2 + \tau^2} \right] \]

\[ \Rightarrow \quad \tau^2 = 135 - 25^2 \]

\[ \Rightarrow \quad \tau = 132.7 N/mm^2 \]

Giving

\[ \sigma_1 = 210 N/mm^2; \quad \sigma_2 = -60 N/mm^2 \]

Case B;
We now have \( \tau_{\text{max}} \) as the radius of the original Mohr's circle linking our stress data.

Thus

\[ \tau_{\text{max}} = \sqrt{25^2 + \tau^2} = 105 \Rightarrow \tau = 101.98 N/mm^2 \]

Principal stresses

\[ \sigma_{1,2} = 75 \pm 105 \quad \Rightarrow \sigma_1 = 180 N/mm^2; \quad \sigma_2 = -30 N/mm^2 \]

Thus Case B applies and the yield stress is 101.98 N/mm\(^2\). The torque required to cause failure is

\[ T = \pi D^2 \tau' / 2 = 32 kNm \]

Failure will occur along a plane at angle \( \beta \) anticlockwise from the y (hoop) direction;

\[ \tan(2\lambda) = \frac{102}{75} \Rightarrow 2\lambda = 76.23^\circ; \]

\[ 2\beta = 90 - 2\lambda \Rightarrow \beta = 6.9^\circ \]
Example More of Von Mises Failure Criterion

From our second Tresca Example

\[ \sigma_h = 100 \text{N/mm}^2 \]
\[ \sigma_L = 50 \text{N/mm}^2 \]
\[ \sigma_r \approx 0 \text{N/mm}^2 \]

What torque will cause failure if the yield stress for steel is 210 N/mm²?

Mohr's Circle

\[ \sigma_1 = s + t = 75 + \sqrt{25^2 + \tau^2} \]
\[ \sigma_2 = s - t = 75 - \sqrt{25^2 + \tau^2} \]
\[ \sigma_3 = 0 \]

At failure
\[ U_D = \frac{1}{12G} \left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\} \]
Or

\[
\begin{align*}
(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 &= (\sigma_y)^2 + (0)^2 + (0 - \sigma_y)^2 \\
4t^2 + (s - t)^2 + (s + t)^2 &= 2\sigma_y^2 \\
2s^2 + 6t^2 &= 2\sigma_y^2 \\
s^2 + 3t^2 &= \sigma_y^2
\end{align*}
\]

\[\Rightarrow 75^2 + 3(25^2 + \tau^2) = 210^2 \quad \tau = 110 \, N/mm^2\]

The tube can thus support a torque of

\[T = \frac{\pi D^2 t' \tau}{2} = 35 \, kNm\]

which is larger than the value of 32 kNm given by Tresca's criterion - in this case, Tresca is more conservative.
7. Strains

7.1. Direct and Shear Strains

Consider a vector of length $l_x$ lying along the x-axis as shown in Figure 1. Let it be subjected to a small strain, so that, if the left hand end is fixed the right hand end will undergo a small displacement $\delta_x$. This need not be in the x-direction and so will have components $\delta_{xx}$ in the x-direction and $\delta_{xy}$ in the y-direction.

![Figure 1](image1)

We can define strains $\varepsilon_{xx}$ and $\varepsilon_{xy}$ by,

$$
\varepsilon_{xx} = \frac{\delta_{xx}}{l_x}; \quad \varepsilon_{xy} = \frac{\delta_{xy}}{l_x}
$$

$\varepsilon_{xx}$ is the direct strain, i.e. the fractional increase in length in the direction of the original vector. $\varepsilon_{xy}$ represents rotation of the vector through the small angle $\gamma_1$ where,

$$
\gamma_1 \equiv \tan \gamma_1 = \frac{\delta_{xy}}{l_x + \delta_{xx}} \approx \frac{\delta_{xy}}{l_x} = \varepsilon_{xy}
$$

Thus in the limit as $\delta_x \to 0$, $\gamma_1 \to \varepsilon_{xy}$.

Similarly we can define strains $\varepsilon_{yy}$ and $\varepsilon_{yx} = \gamma_2$ by,

$$
\varepsilon_{yy} = \frac{\delta_{yy}}{l_y}; \quad \varepsilon_{yx} = \frac{\delta_{yx}}{l_y}
$$
The *ENGINEERING SHEAR STRAIN* is defined as the change in an angle relative to a set of axes originally at 90°. In particular $\gamma_{xy}$ is the change in the angle between lines which were originally in the x- and y-directions. Thus, in our example (Figure 2 above):

$$\gamma_{xy} = (\gamma_1 + \gamma_2) = \varepsilon_{xy} + \varepsilon_{yx} \quad \text{or} \quad \gamma_{xy} = -(\gamma_1 + \gamma_2)$$

depending on sign convention.

Positive values of the shear stresses $\tau_{xy}$ and $\tau_{yx}$ act on an element as shown in Figure 3a and these cause distortion as in Figure 3b. Thus it is sensible to take $\gamma_{xy}$ as +ve when the angle ABC decreases. Thus
\[ \gamma_{xy} = \pm (\gamma_1 + \gamma_2) \]

Or, in general terms:

\[ \gamma_{ij} = (\varepsilon_{ij} + \varepsilon_{ji}) \]

and since

\[ \tau_{ij} = \tau_{ji} \]

we have

\[ \gamma_{ij} = \gamma_{ji} \]

Note that the **TENSOR SHEAR STRAINS** are given by the averaged sum of shear strains:

\[
\frac{1}{2} \gamma_{ij} = \frac{1}{2} (\varepsilon_{ij} + \varepsilon_{ji}) = \frac{1}{2} (\gamma_1 + \gamma_2) = \frac{1}{2} \gamma_{ji}
\]

### 7.2 Mohr’s Circle for Strains

The strain tensor can now be written as:

\[
\varepsilon_{ij} = \begin{bmatrix}
\frac{1}{2} \gamma_{12} & \frac{1}{2} \gamma_{13} \\
\frac{1}{2} \gamma_{21} & \epsilon_{22} & \frac{1}{2} \gamma_{23} \\
\frac{1}{2} \gamma_{31} & \frac{1}{2} \gamma_{32} & \epsilon_{33}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} \gamma_{12} & \frac{1}{2} \gamma_{13} \\
\frac{1}{2} \gamma_{21} & \epsilon_{22} & \frac{1}{2} \gamma_{23} \\
\frac{1}{2} \gamma_{31} & \frac{1}{2} \gamma_{32} & \epsilon_{33}
\end{bmatrix}
\]

where the diagonal elements are the *stretches or tensile strains* and the off diagonal elements are the *tensor shear strains*.

Thus our strain tensor is symmetrical, and:
\[ \varepsilon_{ij} = \varepsilon_{ji} \]

This means there must be a co-ordinate transformation, such that:

\[ \varepsilon_{ij}' \rightarrow \varepsilon_{ij} \]

such that:

\[ \varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \]

we only have principal (=longitudinal) strains!

Exactly analogous to our discussion for the transformation of the stress tensor we find this from:

\[ \varepsilon_{ij}' = a_{ik} \varepsilon_{kl} a_{jl} \]

\[ = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \]

\[ = \begin{pmatrix} \varepsilon_1 \cos^2 \theta + \varepsilon_2 \sin^2 \theta & \varepsilon_1 \cos \theta \sin \theta + \varepsilon_2 \cos \theta \sin \theta & 0 \\ -\varepsilon_1 \cos \theta \sin \theta + \varepsilon_2 \cos \theta \sin \theta & \varepsilon_1 \cos^2 \theta + \varepsilon_2 \sin^2 \theta & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \]

And hence:
\[ \varepsilon_{11}' = \varepsilon_1 \cos^2 \theta + \varepsilon_2 \sin^2 \theta \]

\[ = \frac{1}{2}(\varepsilon_1 + \varepsilon_1) - \frac{1}{2}(\varepsilon_2 - \varepsilon_1) \cos 2\theta \]

\[ \varepsilon_{22}' = \varepsilon_1 \sin^2 \theta + \varepsilon_2 \cos^2 \theta \]

\[ = \frac{1}{2}(\varepsilon_1 + \varepsilon_1) + \frac{1}{2}(\varepsilon_2 - \varepsilon_1) \cos 2\theta \]

\[ \varepsilon_{12}' = \frac{1}{2} \gamma_{12}' = -\varepsilon_1 \cos \theta \sin \theta + \varepsilon_2 \cos \theta \sin \theta \]

\[ = \frac{1}{2}(\varepsilon_2 - \varepsilon_1) \sin 2\theta \]

For which we can draw a Mohr’s circle in the usual manner:
Note, however, that on this occasion we plot half the shear strain against the direct strain. This stems from the fact that the \textit{engineering shear strains} differs from the \textit{tensorial shear strains} by a factor of 2 as discussed.

### 7.3 Measurement of Stress and Strain - Strain Gauges

It is difficult to measure internal stresses. Strains, at least those on a surface, are easy to measure.

- Glue a piece of wire on to a surface
- Strain in wire = strain in material
- As the length of the wire increases, its radius decreases so its electrical resistance increases and can be readily measured.

In practice, multiple wire assemblies are used in strain gauges, to measure direct strains.

Strain rosettes are employed to obtain three measurements:

#### 7.3.1 45° Strain Rosette

Three direct strains are measured

Mohr’s circle for strains gives
radius \( t \)
so we can write

\[
\varepsilon_A = s + t \cos(2\theta)
\]

\[
\varepsilon_B = s + t \cos(2\theta + 90) = s - t \sin(2\theta)
\]

\[
\varepsilon_C = s - t \cos(2\theta)
\]

3 equations in 3 unknowns

Using strain gauges we can find the directions of Principal strains
7.4 Hooke’s Law for Shear Stresses

St. Venant’s Principle states that the principal axes of stress and strain are co-incident. Consider a 2-D element subject to pure shear ($\tau_{xy} = \tau_{yx} = \tau_o$).

The Mohr’s circle for stresses is

where P and Q are principal stress axes and

$$
\sigma_{pp} = \sigma_1 = \tau_o \\
\sigma_{qq} = \sigma_2 = -\tau_o \\
\tau_{pq} = \tau_{qp} = 0
$$

Since the principal stress and strain axes are coincident,

$$
\varepsilon_{pp} = \epsilon_1 = \frac{\sigma_1}{E} - \frac{\nu \sigma_2}{E} = \frac{\tau_o}{E} (1 + \nu) \\
\varepsilon_{qq} = \epsilon_2 = \frac{\sigma_2}{E} - \frac{\nu \sigma_1}{E} = -\frac{\tau_o}{E} (1 + \nu)
$$

and the Mohr’s circle for strain is thus

the Mohr’s circle shows that

$$
\varepsilon_{xx} = 0 \\
-\frac{\gamma_{xy}}{2} = -\frac{\tau_o}{E} (1 + \nu)
$$
So pure shear causes the shear strain $\gamma$

$$\gamma = \frac{2\tau_o}{E} (1 + \nu)$$

But by definition $\tau_o = G\gamma$ so

$$G = \frac{\tau_o}{\gamma} = \frac{E}{2(1 + \nu)}$$

Use St Venants principal to work out principal stress values from a knowledge of principal strains.

Two Mohrs circles, strain and stress.

$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\nu \sigma_2}{E} - \frac{\nu \sigma_3}{E}$$

$$\varepsilon_2 = -\frac{\nu \sigma_1}{E} + \frac{\sigma_2}{E} - \frac{\nu \sigma_3}{E}$$

$$\varepsilon_3 = -\frac{\nu \sigma_1}{E} - \frac{\nu \sigma_2}{E} + \frac{\sigma_3}{E}$$
So using strain gauges you can work out magnitudes of principal strains. You can then work out magnitudes of principal stresses. Using Tresca or Von Mises you can then work out whether your vessel is safe to operate. ie below the yield criteria.